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ОПТИМИЗАЦИЯ РАСПРЕДЕЛЕНИЯ АКТИВОВ В ИНВЕСТИЦИОННОМ ПОРТФЕЛЕ С ИСПОЛЬЗОВАНИЕМ КОВАРИАЦИОННОГО АНАЛИЗА

ASSET ALLOCATION OPTIMIZATION USING COVARIANCE-BASED PORTFOLIO MODELING

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Аннотация. Статья посвящена исследованию возможностей применения модели Марковица при формировании инвестиционного портфеля с минимальным уровнем риска при сохранении требуемой доходности. Теоретической базой работы выступает современная портфельная теория, рассматривающая взаимосвязь между ожидаемой доходностью финансовых инструментов и уровнем принимаемого риска. В качестве эмпирической основы использованы акции компаний, представляющих различные отрасли экономики, что позволило обеспечить необходимую степень диверсификации исследуемого портфеля. В ходе исследования выполнен расчет логарифмических доходностей, показателей волатильности и ковариационной матрицы доходностей активов. На основании полученных результатов проведена оценка характеристик равновзвешенного портфеля, после чего осуществлена процедура его оптимизации с применением модели среднее–дисперсия и методов условной оптимизации. Для численного решения поставленной задачи использовались вычислительные инструменты Microsoft Excel. Полученные результаты свидетельствуют о возможности существенного снижения совокупного инвестиционного риска без изменения целевого уровня доходности. Проведенный анализ подтверждает высокую практическую значимость математических методов в управлении инвестициями и демонстрирует эффективность использования модели Марковица при принятии решений на финансовых рынках.

Abstract. This study investigates the use of the Markowitz portfolio optimization model to construct an investment portfolio with the lowest possible level of risk while maintaining a predefined return target. The research is grounded in Modern Portfolio Theory, which examines the relationship between expected returns and the risk associated with financial assets. To ensure an adequate degree of diversification, the analysis is based on stocks representing different sectors of the economy. The study includes the calculation of logarithmic returns, volatility indicators, and the covariance matrix of asset returns. Using the obtained data, an equally weighted portfolio was evaluated and subsequently optimized through the mean-variance framework and constrained optimization techniques. Numerical computations were performed using Microsoft Excel. The findings indicate that portfolio risk can be significantly reduced without compromising the desired level of expected return. The results confirm the practical value of mathematical methods in investment management and highlight the effectiveness of the Markowitz approach for portfolio decision-making in financial markets.

Ключевые слова: модель Марковица, инвестиционный портфель, оптимизация, риск, доходность, диверсификация, ковариация, финансовая математика

Keywords: Markowitz model, investment portfolio, optimization, risk, return, diversification, covariance, financial mathematics

Introduction

In the context of modern financial markets, the efficient allocation of capital among available financial assets remains one of the central challenges of investment analysis. Investors continuously seek to maximize portfolio returns while maintaining risk within acceptable limits, which necessitates the application of quantitative analytical techniques and optimization methods. Among the most influential concepts in contemporary financial theory is the portfolio selection framework introduced by Harry Markowitz, which is based on the assessment of the relationship between expected return and investment risk [1].

The Markowitz framework laid the foundation for Modern Portfolio Theory and significantly contributed to the development of quantitative approaches to investment management. Within this paradigm, investment risk is interpreted as the statistical variability of asset returns and is commonly measured through variance and standard deviation indicators [2]. A fundamental component of the model is diversification, which enables investors to reduce overall portfolio risk by combining assets whose returns exhibit different correlation patterns [3].

Recent advances in financial mathematics have further demonstrated that asset allocation decisions play a critical role in determining the effectiveness of investment strategies [4]. By incorporating covariance relationships among asset returns and applying mathematical optimization techniques, it becomes possible to identify portfolio compositions that minimize risk while satisfying a predetermined return objective [1, 2]. One of the most widely used analytical tools for solving such constrained optimization problems is the method of Lagrange multipliers, which allows optimal solutions to be obtained under a set of specified restrictions [5].

Although more sophisticated portfolio models have emerged over recent decades, the Markowitz approach continues to be extensively employed in both academic research and practical portfolio management [6]. Its enduring relevance can be attributed to its analytical transparency, strong theoretical foundation, and adaptability to a broad range of financial instruments and market conditions [7].

This study investigates the optimization of an investment portfolio composed of equities representing different sectors of the economy. The analysis is conducted using historical return data within the mean–variance framework proposed by Markowitz. The research involves the estimation of expected returns, standard deviations, and the covariance matrix of asset returns, followed by the determination of the portfolio allocation that achieves the minimum attainable level of risk for a specified target return [8].

The primary objective of this research is to evaluate the effectiveness of the Markowitz model as a tool for portfolio risk minimization. To achieve this objective, the performance of an equally weighted portfolio is compared with that of an optimized portfolio obtained through mathematical optimization procedures, allowing the benefits of optimal asset allocation to be quantitatively assessed [9].

1. Data Collection.

Diversification represents one of the fundamental principles of the Markowitz portfolio framework, requiring the allocation of investment capital across assets originating from different sectors of the economy [1]. Such an approach contributes to risk reduction by decreasing the dependence between the return dynamics of individual financial instruments and enhancing the overall stability of portfolio performance [2].

For the purposes of this study, historical monthly closing price data were collected from the financial information platforms Yahoo Finance and Investing.com. The selected sample included shares of Tesla (automotive industry), Berkshire Hathaway (financial services sector), Amazon (e-commerce industry), and Vertex Pharmaceuticals (pharmaceutical sector). Although these companies may be influenced by common macroeconomic and market factors, they operate in distinct industries and are not direct competitors, thereby providing an appropriate basis for portfolio diversification in accordance with the principles of Modern Portfolio Theory [3].

The analysis covers the period from October 2024 to September 2025, corresponding to the most recent completed U.S. financial year. To ensure consistency and readability, all numerical values reported in the study were rounded to two decimal places.

Figure 1 presents the initial dataset consisting of the monthly closing stock prices of the selected companies. These observations served as the basis for the subsequent estimation of asset returns, volatility measures, and portfolio risk characteristics.

	A	B	C	D	E
1	Date	Tesla	Berkshire A	Amazon	Vertex
2	01.09.2024	261,63	691,180	186,33	465,08
3	01.10.2024	249,85	676,96	186,4	475,98
4	01.11.2024	345,16	724,04	207,89	468,13
5	01.12.2024	403,84	680,92	219,39	402,7
6	01.01.2025	404,6	702,61	237,68	461,68
7	01.02.2025	292,98	775,00	212,28	479,79
8	01.03.2025	259,16	798,44	190,26	484,82
9	01.04.2025	282,16	800,54	184,42	509,5
10	01.05.2025	346,46	757,40	205,01	442,05
11	01.06.2025	317,66	728,80	219,39	445,2
12	01.07.2025	308,27	719,85	234,11	456,87
13	01.08.2025	333,87	755,28	229	391,02
14	01.09.2025	444,72	754,20	219,57	391,64

Figure 1 - Monthly closing stock prices of Tesla, Berkshire Hathaway, Amazon, and Vertex Pharmaceuticals for the period from October 2024 to September 2025.

2. Estimation of Expected Returns.

The return of each asset was calculated using logarithmic returns, which are based on the natural logarithm of the relative change in the asset price. Compared with simple arithmetic returns, logarithmic returns provide a more robust framework for financial analysis due to their time-additive properties and their ability to capture proportional price changes more accurately over multiple periods [6]. Consequently, log returns are extensively employed in financial economics, risk management, and quantitative portfolio modeling [10].

Monthly returns were computed using the following expression:

$$r_i = \ln\left(\frac{P_t}{P_{t-1}}\right) \qquad (1)$$

where:

r_i - return of the asset during the given period;

P_t - closing price of the stock in the current month;

P_{t-1} - closing price of the stock in the preceding month.

Thus, the monthly return of Tesla stock for the first observation interval (October–November) was obtained by substituting the corresponding closing prices into the above equation. For all subsequent periods, the calculation procedure remained unchanged, with the current month's closing price serving as P_t, and the previous month's closing price serving as P_{t-1}. The same methodology was consistently applied to all securities included in the analyzed investment portfolio.

The expected monthly return of each asset was then estimated as the arithmetic mean of the calculated periodic returns over the entire observation horizon [7]. The general expression for expected return is given by:

$$\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i \qquad (2)$$

where:

$\bar{r}$ - expected return of the asset;

r_i - return observed during a specific period;

n - total number of observations included in the analysis.

Substituting the calculated monthly returns of Tesla into the above equation yields the expected monthly return of the asset:

$$\bar{r} = \frac{1}{12}\sum_{i=1}^{12} r_i = \frac{-4.61 + 15.7 + \cdots + 28.67}{12} = \frac{53.05}{12} \approx 4.42$$

Following the substitution of Tesla's monthly return values into the expected return formula, the average monthly return of the asset was obtained. The same procedure was subsequently applied to the remaining securities included in the investment portfolio. As a result, the first fundamental parameter of the Markowitz framework—expected asset return—was determined for each stock under consideration [1].

It is noteworthy that the estimated expected return of Vertex Pharmaceuticals was found to be negative over the analyzed period. From a conventional investment perspective, such a result may suggest that including this asset in the portfolio is economically unjustified. However, this outcome also illustrates one of the inherent limitations of the Markowitz model, namely its reliance on historical observations as the primary source of information. Since financial markets are influenced by a wide range of macroeconomic, industry-specific, and firm-level factors, historical performance does not necessarily provide a reliable indicator of future returns [4]. This limitation has also been extensively discussed within the framework of the Efficient Market Hypothesis, which emphasizes the uncertainty associated with forecasting future market behavior based solely on past data [11].

Nevertheless, a negative expected return does not automatically imply that an asset should be excluded from portfolio construction. Within the context of portfolio diversification, securities exhibiting weak correlations with other portfolio components may contribute to a reduction in overall portfolio volatility. Consequently, even assets with relatively unfavorable return characteristics can improve the risk profile of a diversified portfolio when their return dynamics differ substantially from those of other financial instruments [2].

3. Estimation of Standard Deviation.

The second fundamental parameter within the Markowitz framework is risk, which is quantified through the sample standard deviation of asset returns. The use of the sample standard deviation is justified by the fact that the analysis relies on historical observations and aims to estimate the potential variability of future asset performance rather than merely describe past outcomes [9].

Standard deviation measures the dispersion of returns around their mean value and therefore serves as an indicator of asset volatility [2]. Within the context of Modern Portfolio Theory, volatility is widely regarded as the primary quantitative measure of investment risk, reflecting the degree of uncertainty associated with future returns [1]. Higher levels of volatility imply greater fluctuations in asset prices and, consequently, a higher degree of investment risk [12].

The standard deviation of an individual stock's return is calculated using the following expression:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(r_i - \bar{r})^2}{n-1}} \qquad (3)$$

where:

σ - standard deviation of the asset's returns;

r_i - return observed during a specific period;

$\bar{r}$ - expected return of the asset;

n - number of observations included in the sample.

By substituting the observed return values for Tesla into the above equation, the volatility of the asset can be estimated through its standard deviation. This measure provides a quantitative assessment of the asset's risk by capturing the extent to which individual monthly returns deviate from their expected value:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{12}(r_i - 4.42)^2}{12-1}}$$

$$= \sqrt{\frac{\left((-4.61) - 4.42\right)^2 + (32.31 - 4.42)^2 + \cdots + (28.67 - 4.42)^2}{11}}$$

$$= \sqrt{\frac{(-9.03)^2 + (27.89)^2 + \cdots + (24.25)^2}{11}}$$

$$= \sqrt{\frac{81.5409 + 777.8521 + \cdots + 588.0625}{11}} \approx 19.09$$

It should be noted that the mean value appearing in the standard deviation formula corresponds to the expected return of the asset. Consequently, the calculated risk measure reflects the extent to which actual periodic returns deviate from their average level. The greater the magnitude of the standard deviation, the more volatile the asset is considered to be and, therefore, the higher the degree of investment risk associated with holding that security [10].

The same calculation procedure was applied to all remaining stocks included in the investment portfolio. The resulting estimates of expected returns and standard deviations are summarized in the Excel worksheet presented in Figure 2.

Date	Tesla	Berkshire A	Amazon	Vertex	Tesla	Berkshire	Amazon	Vertex
01.09.2024	261,63	691,180	186,33	465,08				
01.10.2024	249,85	676,96	186,4	475,98	-4,61%	-2,08%	0,04%	2,32%
01.11.2024	345,16	724,04	207,89	468,13	32,31%	6,72%	10,91%	-1,66%
01.12.2024	403,84	680,92	219,39	402,7	15,70%	-6,14%	5,38%	-15,06%
01.01.2025	404,6	702,61	237,68	461,68	0,19%	3,14%	8,01%	13,67%
01.02.2025	292,98	775,00	212,28	479,79	-32,28%	9,81%	-11,30%	3,85%
01.03.2025	259,16	798,44	190,26	484,82	-12,27%	2,98%	-10,95%	1,04%
01.04.2025	282,16	800,54	184,42	509,5	8,50%	0,26%	-3,12%	4,97%
01.05.2025	346,46	757,40	205,01	442,05	20,53%	-5,54%	10,58%	-14,20%
01.06.2025	317,66	728,80	219,39	445,2	-8,68%	-3,85%	6,78%	0,71%
01.07.2025	308,27	719,85	234,11	456,87	-3,00%	-1,24%	6,49%	2,59%
01.08.2025	333,87	755,28	229	391,02	7,98%	4,80%	-2,21%	-15,56%
01.09.2025	444,72	754,20	219,57	391,64	28,67%	-0,14%	-4,21%	0,16%
			Expected return	(Monthly)	4,42%	0,73%	1,37%	-1,43%
				Risk	19,09%	5,07%	8,17%	9,35%

Figure 2 - Expected return and standard deviation estimates for the selected stocks included in the investment portfolio.

4. Construction of the Covariance Matrix.

While expected returns are required to estimate the overall profitability of an investment portfolio, standard deviations provide a measure of the risk associated with individual assets. However, assessing portfolio risk requires more than evaluating the volatility of each security in isolation. It is also necessary to examine the relationships among asset returns, as interactions between securities can significantly influence the overall risk profile of the portfolio [2].

To capture these relationships, a covariance matrix was constructed containing covariance estimates for every pair of assets included in the analysis. Covariance measures both the direction and the degree of joint variation between asset returns.

A positive covariance indicates that two assets tend to move in the same direction, whereas a negative covariance suggests an inverse relationship between their return dynamics [6]. The incorporation of covariance effects represents one of the principal advantages of the Markowitz framework, as it allows portfolio risk to be evaluated from a multidimensional perspective rather than solely through the volatility of individual assets [1].

In modern financial theory, covariance analysis is widely recognized as a fundamental component of quantitative risk management and portfolio construction methodologies [13]. The general formula used to calculate covariance is given as follows:

$$Cov(X,Y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1} \qquad (4)$$

where:

$Cov(X,Y)$ - covariance between the returns of assets X and Y;

x_i, y_i - observed returns of the respective assets during period i;

$\bar{x}, \bar{y}$ - mean returns of assets X and Y;

n - number of observations included in the sample.

In this study, the sample covariance estimator was employed because the analysis is based on historical observations, which represent only a subset of all possible market outcomes [5]. The use of sample covariance provides an unbiased estimate of the underlying population parameters and is therefore more appropriate for empirical financial analysis.

To ensure dimensional consistency throughout the calculations, all percentage returns were converted into decimal form prior to the estimation process. Accordingly, a return of 1% was represented as 0.01, and all monthly return observations were divided by 100 before computing covariance values. This approach is consistent with standard practices in financial modeling and is widely adopted in empirical studies of financial markets and econometric analysis [8, 14].

To illustrate the calculation procedure, the covariance between the returns of Tesla and Berkshire Hathaway was estimated by substituting the corresponding return observations into the covariance formula:

$$Cov(T,B) = \frac{\sum_{i=1}^{12}(x_i - 0.042)(y_i - 0.0073)}{12 - 1}$$

$$= \frac{\big((-0.046) - 0.042\big)\big((-0.0208) - 0.0073\big) + \cdots + (0.2867 - 0.042)((-0.0014) - 0073)}{11}$$

$$\approx \frac{(-0.03)}{11} \approx -0.00273$$

The same procedure was repeated for all remaining pairs of assets included in the portfolio. In addition, covariance values were calculated for each asset with itself, which correspond to the variances of the respective securities. These variance terms form the diagonal elements of the covariance matrix and represent the individual contribution of each asset to overall portfolio risk.

The complete set of covariance estimates was subsequently organized into a covariance matrix, which serves as a fundamental input for portfolio optimization and risk assessment within the Markowitz framework. The resulting covariance matrix is presented in Figure 3.

portion(w)	**covariance matrix**				
		Tesla	Berkshire	Amazon	Vertex
0,25	Tesla	0,033941	-0,00273	0,00763	-0,006746
0,25	Berkshire	-0,00273	0,002418	-0,00176	0,0015626
0,25	Amazon	0,00763	-0,00176	0,006089	-0,001196
0,25	Vertex	-0,00675	0,001563	-0,0012	0,0080788

Figure 3 - Covariance matrix of stock returns for Tesla, Berkshire Hathaway, Amazon, and Vertex Pharmaceuticals over the study period.

5. Estimation of the Return and Risk of the 1/N Portfolio.

Once the expected returns and the covariance matrix have been determined, it becomes possible to evaluate the overall return and risk characteristics of the investment portfolio. As an initial benchmark, a naïve diversification strategy based on the 1/N allocation rule is considered, where investment capital is distributed equally among all available assets [2]. Due to its simplicity and ease of implementation, the equally weighted portfolio is frequently used as a reference point when assessing the effectiveness of more sophisticated portfolio optimization techniques [15].

The expected return of the portfolio is calculated as the weighted average of the expected returns of its constituent assets. In general form, the portfolio return can be expressed as follows:

$$R_p = \sum_{i=1}^{n} \omega_i r_i \qquad (5)$$

where:

R_p - expected return of the portfolio;

ω_i - weight of asset i in the portfolio;

r_i - expected return of asset i.

In the present study, the portfolio consists of four stocks. Under the equally weighted 1/N allocation strategy, each asset receives an identical proportion of the total investment capital. Therefore, the portfolio weights are defined as:

$$\omega_i = 0.25$$

for all assets included in the portfolio.

Substituting the previously estimated expected returns into the portfolio return equation yields the expected return of the equally weighted portfolio:

$$R_p = \sum_{i=1}^{4} 0.25 \cdot r_i = 4.42 \cdot 0.25 + 0.73 \cdot 0.25 + 1.37 \cdot 0.25 - 1.43 \cdot 0.25 \approx 1.27\%$$

The overall risk of a portfolio is evaluated using the standard deviation framework while incorporating the covariance relationships among all constituent assets. Unlike expected return, portfolio risk is determined not only by the individual volatility of each security but also by the degree to which asset returns move together over time [1].

Consequently, the risk of a diversified portfolio depends on both the variances of individual assets and the covariance structure of the portfolio as a whole. This feature represents one of the key insights of Modern Portfolio Theory, demonstrating that portfolio risk cannot be assessed solely by examining assets independently.

The general expression for portfolio risk is given by:

$$\sigma_p = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i\omega_j\,Cov(r_i, r_j)} \qquad (6)$$

For a portfolio consisting of four assets, the portfolio standard deviation can be expressed in expanded form as:

$$\sigma_p = \sqrt{\omega_1^2 cov_{11} + \omega_2^2 cov_{22} + \omega_3^2 cov_{33} + \omega_4^2 cov_{44} + 2(\omega_1\omega_2 cov_{12} + \cdots + \omega_3\omega_4 cov_{34})}$$

This formulation incorporates both the variances of the individual assets and the covariance terms associated with every pair of securities included in the portfolio. Consequently, portfolio volatility is influenced not only by the risk characteristics of individual assets but also by the degree of interaction among their returns. Such an approach provides a more comprehensive assessment of investment risk than analyzing assets separately.

By substituting the previously estimated variances, covariances, and portfolio weights into the above equation, the overall volatility of the equally weighted portfolio was obtained.

The results indicate that the expected return of the 1/N portfolio equals 1.27%, while its overall risk, measured by portfolio standard deviation, amounts to 5.25%. These findings illustrate the risk–return characteristics of an equally weighted allocation strategy and provide a benchmark against which the performance of optimized portfolios can be evaluated [7].

6. Application of the Lagrange Multiplier Method.

The next stage of the study involves optimizing the portfolio structure through adjustments to the asset allocation weights. The primary objective of this procedure is to minimize the overall portfolio risk while maintaining a predetermined level of expected return [5]. Among the most widely used mathematical approaches for solving such constrained optimization problems is the method of Lagrange multipliers, which enables the identification of optimal solutions subject to a set of specified constraints [5].

In the present research, the Lagrange multiplier method is employed to determine the portfolio composition that achieves the lowest attainable level of risk while preserving a fixed expected return. Although the method is highly effective, its analytical implementation becomes increasingly complex as the number of decision variables grows. Therefore, to illustrate the mathematical foundations of the approach, a simplified portfolio consisting of two assets—Tesla and Berkshire Hathaway—is considered. The optimization of the complete four-asset portfolio will subsequently be performed using Microsoft Excel optimization tools [8].

The optimization objective is to achieve an expected portfolio return of 1.27%, corresponding to the return generated by the equally weighted 1/N portfolio, while simultaneously minimizing portfolio risk. Prior to constructing the Lagrangian function, it is necessary to define the set of constraints governing the optimization problem. The first constraint requires that the sum of all portfolio weights equals one, ensuring that the entire investment budget is allocated across the available assets. The second constraint imposes a fixed target level of expected portfolio return [6].

The full-investment constraint can be written as follows:

$$\omega_1 + \omega_2 = 1$$

The target return constraint is expressed as:

$$\omega_1 r_1 + \omega_2 r_2 = 0.0127$$

where:

ω_1, ω_2 - portfolio weights assigned to Tesla and Berkshire Hathaway, respectively;

r_1, r_2 - expected returns of the corresponding assets.

Based on the calculations presented in the previous sections, the expected monthly return of Tesla was estimated at 4.42% (0.0442), whereas Berkshire Hathaway exhibited an expected return of 0.73% (0.0073). The covariance matrix required for the optimization procedure had also been obtained during the earlier stage of the analysis.

The problem of minimizing portfolio risk subject to a predetermined return level can be formulated using the Lagrangian optimization framework. Within the context of Modern Portfolio Theory, the objective is to determine the vector of asset weights that minimizes portfolio variance while satisfying both the return and budget constraints. For a portfolio consisting of n assets, the general risk-minimization problem can be expressed as follows [5]:

$$L(\omega, \lambda_1, \lambda_2) = \sigma_p^2 + \lambda_1\left(\sum_{i=1}^{n} \omega_i - 1\right) + \lambda_2\left(\sum_{i=1}^{n} \omega_i r_i - R\right) \qquad (7)$$

Based on the general optimization framework, the Lagrangian function for the two-asset portfolio can be formulated as follows:

$$\begin{aligned} L(\omega_1, \omega_2, \lambda_1, \lambda_2) \\ &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2 cov(r_1, r_2) + \lambda_1(\omega_1 + \omega_2 - 1) \\ &+ \lambda_2(\omega_1 r_1 + \omega_2 r_2 - 0.0127) \end{aligned}$$

The first three terms represent the variance of the portfolio, which serves as the objective function to be minimized. The remaining terms incorporate the optimization constraints through the Lagrange multipliers λ_1 and λ_2, ensuring compliance with both the full-investment requirement and the target return condition.

After substituting the estimated asset returns and covariance parameters into the above expression, a specific Lagrangian function was obtained for the portfolio under consideration. The next stage of the optimization procedure involved deriving the first-order conditions by calculating the partial derivatives of the Lagrangian with respect to all decision variables and setting them equal to zero [5].

The necessary conditions for an optimum are therefore given by the following system of equations:

$$\frac{\partial L}{\partial \omega_1} = 0, \qquad \frac{\partial L}{\partial \omega_2} = 0, \qquad \frac{\partial L}{\partial \lambda_1} = 0, \qquad \frac{\partial L}{\partial \lambda_2} = 0$$

The resulting system of equations was solved using Microsoft Excel, which provided an efficient numerical framework for handling the constrained optimization problem. The optimization results indicate that the optimal allocation assigns 15.13% of the portfolio to Tesla and 84.87% to Berkshire Hathaway. For computational convenience in subsequent calculations, these weights were rounded to 0.15 and 0.85, respectively.

Substituting the optimized portfolio weights into the portfolio risk equation yielded a minimum portfolio risk of 4.26% while maintaining the target expected return of 1.27%. This outcome demonstrates that the application of the Lagrange multiplier method leads to a substantial improvement in the portfolio's risk profile compared with the equally weighted 1/N portfolio, whose risk was previously estimated at 5.25%.

The reduction in portfolio volatility, achieved without sacrificing expected return, highlights the practical value of mathematical optimization techniques in investment management. These findings provide empirical support for the effectiveness of mean–variance optimization and confirm that an appropriate allocation of capital can significantly enhance portfolio efficiency by improving the trade-off between risk and return [4].

7. Portfolio Optimization.

Following the analysis of the simplified two-asset case, the optimization procedure was extended to the complete investment portfolio consisting of four stocks. Due to the increased number of decision variables and constraints, the optimization problem was solved using Microsoft Excel, which provides efficient computational tools for handling complex constrained optimization tasks [8].

The objective of the optimization was to identify the portfolio allocation that preserves the target expected return of 1.27% while minimizing overall portfolio risk. The optimization process utilized the expected returns, standard deviations, and covariance matrix estimated in the previous stages of the study as input parameters [1].

The analysis was conducted within the framework of the Markowitz mean–variance model, which seeks to determine the optimal balance between expected return and investment risk [4]. Numerical optimization was performed using the Excel Solver add-in, a widely used tool for solving constrained mathematical programming problems in finance and operations research [8].

The optimization results obtained for the four-asset portfolio are presented in Figure 4.

portion(w)	**covariance matrix**				
		Tesla	Berkshire	Amazon	Vertex
0,12	Tesla	0,033941	-0,00273	0,0076304	-0,006746
0,88	Berkshire	-0,00273	0,002418	-0,0017602	0,0015626
0,06	Amazon	0,00763	-0,00176	0,0060892	-0,001196
0,00	Vertex	-0,00675	0,001563	-0,001196	0,0080788
portion(w)		0,12	0,88	0,06	0,00
Total reurn		1,27%		Sheme 1/n	1,27%
Total risk		4,22%			5,25%
Budget		1			

Figure 4 - Portfolio optimization results obtained using the Microsoft Excel Solver optimization tool.

The application of computational optimization techniques resulted in a reduction of overall portfolio risk by 1.03 percentage points while maintaining the target expected return of 1.27%. These findings demonstrate that an optimal allocation of capital across assets can significantly improve portfolio efficiency relative to the equally weighted 1/N strategy [15].

An important observation is that Vertex Pharmaceuticals was effectively excluded from the optimized portfolio. This outcome can be attributed to the asset’s negative expected return during the analyzed period, which adversely affected the portfolio’s risk–return profile. As a result, the optimization procedure assigned either a negligible or zero weight to this security in favor of assets offering more favorable risk-adjusted performance.

Overall, the implementation of numerical optimization methods led to the construction of a more efficient portfolio characterized by lower risk while preserving the desired level of expected return. These results highlight the practical relevance of mathematical optimization techniques and computational tools in investment decision-making and portfolio management, demonstrating their ability to enhance portfolio performance through informed asset allocation [7].

Conclusion

This study examined the applicability of the Markowitz model for investment portfolio optimization under the objective of risk minimization. The analysis was based on historical closing price data for Tesla, Berkshire Hathaway, Amazon, and Vertex Pharmaceuticals, representing different sectors of the economy. The selection of assets from diverse industries enabled the construction of a sufficiently diversified portfolio and provided a suitable framework for investigating the relationship between expected return and investment risk.

Throughout the study, logarithmic returns, expected returns, standard deviations, and the covariance matrix of asset returns were calculated. The results confirmed that portfolio risk is influenced not only by the volatility of individual assets but also by the interaction among their return dynamics, highlighting the importance of diversification in portfolio construction.

As an initial benchmark, an equally weighted 1/N portfolio was analyzed, with capital allocated uniformly across all assets. The calculations indicated that this portfolio generated an expected return of 1.27% while exhibiting an overall risk level of 5.25%.

Subsequently, portfolio optimization was performed using the Lagrange multiplier method and Microsoft Excel Solver. The optimization process successfully maintained the target expected return of 1.27% while reducing portfolio risk to 4.26%. As a result, the optimized portfolio achieved a risk reduction of 1.03 percentage points relative to the equally weighted allocation strategy.

The findings demonstrate that the Markowitz framework can significantly improve portfolio efficiency through the optimal allocation of capital among available assets. The analysis further revealed that securities with negative expected returns, such as Vertex Pharmaceuticals during the examined period, may be excluded from the optimal portfolio when their inclusion deteriorates the overall risk–return trade-off.

At the same time, several limitations of the model were identified. The primary drawback arises from its reliance on historical data, which may not accurately reflect future market conditions. Financial markets are influenced by numerous macroeconomic, political, and firm-specific factors that are not explicitly incorporated into the classical Markowitz framework. Consequently, historical return patterns cannot guarantee similar outcomes in future periods.

Overall, the results of the study confirm the practical relevance of quantitative methods in investment management. The integration of the Markowitz model, mathematical optimization techniques, and computational tools provides an effective framework for portfolio construction, enabling investors to reduce risk while maintaining a desired level of expected return.

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